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A hierarchy of systems of non-linear equations

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Abstract. Imposing isospectral invariance for the one-dimensional Dirac operator yields an infinite hierarchy of systems of chiral invariant non-linear partial differential equations. The same system is obtained through a Lax pair construction and finally a formulation in terms of Kac-Moody generators is given.

1. Introduction

The first successful solution of the KdV equation led to the development of the inverse scattering transform method. With the help of the construction of Lax pairs or following the AKNS ideas it was possible to generate a large number of solvable non-linear partial differential equations [1, 2]. The next big step by Zhakarov and Shabat [3] showed the connection of the inverse method to the Riemann transform method [4].

Already at the beginning of the development of the KdV equation it was realised that the eigenvalues of the associated Schrödinger operator are left invariant if the potential evolves according to the non-linear equation. Vice versa, one may generate an infinite number of higher-order KdV equations by imposing isospectral invariance to the Schrödinger operator.

In this paper we start from the one-dimensional most general self-adjoint Dirac operator on the full line and ask for systems of non-linear evolution equations for the potentials such that the spectrum of the Dirac operator is left invariant. A simple method, using Feynman-Hellman theorem and an expansion in the spectral parameter, allows us to characterise these systems iteratively (theorem 1). Generalisations of the non-linear Schrödinger equation and of the modified KdV equation belong to them. Next we observe that the same systems can be generated following the AKNS scheme. We show that the action of gauge and chiral transformations for the Dirac operator extend naturally to the non-linear systems. Finally, we remark that a third formulation in terms of generators of a Kac-Moody algebra exists. The use of such infinite algebras may shed new light onto infinite-dimensional integrable Hamiltonian systems.

All these systems can be solved in a certain gauge by following standard inverse scattering transform methods. Soliton solutions for the modified KdV equation and the corresponding $N=3$ system (see equation (19)) have been studied recently in connection with a solid state physics model [5-7].

2. Isospectral flow for the Dirac equation

Our first derivation of the infinite hierarchy of non-linear systems follows from imposing isospectral invariance for the one-dimensional Dirac operator with potentials a_1, c_1

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and $\boldsymbol{v} = (v_1, v_2)$:

$$-\mathrm{i}H(\partial_x - \mathrm{i}a_1 + \boldsymbol{E}\cdot\boldsymbol{v})\psi = (\lambda - c_1)\psi \tag{1}$$

where $\boldsymbol{E}\cdot\boldsymbol{v} = E^1 v_1 + E^2 v_2$ and H, E^1 and E^2 denote generators of sl(2, C) satisfying

$$[H, E^{\overset{1}{2}}] = \pm 2\mathrm{i}E^{\overset{2}{1}} \qquad [E^{\overset{1}{2}}, E^{\overset{1}{2}}] = 0 \qquad [E^1, E^2] = 2\mathrm{i}H \tag{2}$$

which may be expressed by Pauli matrices. We assume that all four potentials and the spinor ψ depend on an additional parameter which we may call time t.

For the discrete spectrum of (1) we impose the condition that it remains time independent. Let ε be an eigenvalue with eigenfunction ψ. From the Feynman–Hellman theorem we deduce that

$$\partial_t\varepsilon = \int_{-\infty}^{\infty} \mathrm{d}x\{I^a\partial_t a_1 + I^c\partial_t c_1 + \boldsymbol{I}^v\cdot\partial_t\boldsymbol{v}\} \tag{3}$$

where we introduced densities

$$I^a = -\psi^\dagger H\psi \qquad I^c = \psi^\dagger\psi \qquad \boldsymbol{I}^v = -\mathrm{i}\psi^\dagger H\boldsymbol{E}\psi. \tag{4}$$

Now we ask for a time evolution of the potentials such that $\partial_t\varepsilon = 0$. The ansatz for

$$\partial_t\varepsilon = \int_{-\infty}^{\infty} \mathrm{d}x\,\partial_x(a_2 I^a + c_2 I^c + \boldsymbol{b}\cdot\boldsymbol{I}^v) \tag{5}$$

as a total differential leads, after using (1) and comparing (3) with (5), to the relations

$$\begin{aligned} &\partial_t a_1 = \partial_x a_2 \\ &\partial_t c_1 = \partial_x c_2 - 2\mathrm{i}\boldsymbol{b}\cdot\sigma_2\boldsymbol{v} \\ &\partial_t\boldsymbol{v} = -2\mathrm{i}c_2\sigma_2\boldsymbol{v} + \partial_x\boldsymbol{b} + 2\mathrm{i}(c_1 - \varepsilon)\sigma_2\boldsymbol{b} \end{aligned} \tag{6}$$

where the Pauli matrix σ_2 enters via the structure constants of sl(2, C). Next we expand a_2, $\boldsymbol{b}$ and c_2 in powers of ε:

$$a_2 = \sum_{n=0}^{N} \varepsilon^n a_2^{(n)} \qquad \boldsymbol{b} = \sum_{n=0}^{N} \varepsilon^n \boldsymbol{b}^{(n)} \qquad c_2 = \sum_{n=0}^{N} \varepsilon^n c_2^{(n)} \tag{7}$$

where the expansion coefficients $a_2^{(n)}$, $\boldsymbol{b}^{(n)}$ and $c_2^{(n)}$ depend on all potentials and their derivatives. Introducing (7) into (6) leads to the recurrence relations

$$\begin{aligned} &\partial_x a_2^{(n)} = 0 && 1 \leqslant n \leqslant N \\ &\boldsymbol{b}^{(N)} = 0 \qquad 2\boldsymbol{b}^{(n)} = -2c_2^{(n+1)}\boldsymbol{v} - \mathrm{i}\sigma_2 D_x\boldsymbol{b}^{(n+1)} && 0 \leqslant n < N \\ &\partial_x c_2^{(N)} = 0 \qquad \partial_x c_2^{(n)} = 2\mathrm{i}\boldsymbol{b}^{(n)}\cdot\sigma_2\boldsymbol{v} && 1 \leqslant n < N \end{aligned} \tag{8}$$

together with time evolution equations for the potentials:

$$\begin{aligned} &\partial_t a_1 = \partial_x a_2^{(0)} \\ &\partial_t c_1 = \partial_x c_2^{(0)} - 2\mathrm{i}\boldsymbol{b}^{(0)}\cdot\sigma_2\boldsymbol{v} \\ &D_t\boldsymbol{v} = D_x\boldsymbol{b}^{(0)} \end{aligned} \tag{9}$$

where D_t and D_x denote covariant derivatives

$$D_t = \partial_t + 2\mathrm{i}\sigma_2 c_2^{(0)} \qquad D_x = \partial_x + 2\mathrm{i}\sigma_2 c_1. \tag{10}$$

Remark. One of the fields c_1 or $c_2^{(0)}$ may be chosen arbitrarily as a function of t and x. The same holds for a_1 and $a_2^{(0)}$; these two even obey evolution equations which decouple completely from the other equations in (9).

Remark on gauge invariance. It was realised some time ago that certain non-linear equations are gauge equivalent to each other [8].

Here we note that the invariance of the Dirac equation (1) under gauge and chiral transformations

$$\begin{array}{ll} \psi \rightarrow \exp(\mathrm{i}\Lambda - \mathrm{i}H\chi)\psi & \boldsymbol{v} \rightarrow \exp(-2\mathrm{i}\sigma_2\chi)\boldsymbol{v} \\ a_1 \rightarrow a_1 + \partial_x\Lambda & c_1 \rightarrow c_1 + \partial_x\chi \end{array} \tag{11}$$

implies gauge invariance of (9). Under the transformation (11) together with

$$a_2^{(0)} \rightarrow a_2^{(0)} + \partial_t\Lambda \qquad c_2^{(0)} \rightarrow c_2^{(0)} + \partial_t\chi \tag{12}$$

the systems (8) and (9) remain invariant. Clearly, only the chiral part of the transformation of ψ has non-trivial consequences for the system.

In an attempt to generate systems of the hierarchy from equation (8) one fixes N and is faced with the problem of integrating the equation for $\partial_x c_2^{(n)}$ for $n = N-1, N-2, \ldots, 1$. It turns out that the RHS for this quantity is always a total derivative which allows us to formulate the following theorem.

Theorem 1. The iteration procedure (equations (8) and (9)) determines systems of non-linear equations for the potentials of the Dirac equation leaving its spectrum invariant. All expansion coefficients in equations (7) are polynomials in the potentials and their derivatives.

Proof. We will show by complete induction that $\partial_x c_2^{(n)}$ is a total derivative for all n; without loss of generality we put

$$c_2^{(N)} = 1 \qquad \boldsymbol{b}^{(N-1)} = -\boldsymbol{v}.$$

Suppose the assertion is true for $\partial_x c_2^{(j)}$, $n+1 \leqslant j \leqslant N$; we will show that $\partial_x c_2^{(n)}$ is a total derivative in two steps.

(*a*) $$\boldsymbol{b}^{(m)} \cdot \sigma_2 \boldsymbol{b}^{(r)} = \boldsymbol{b}^{(m-1)} \cdot \sigma_2 \boldsymbol{b}^{(r+1)} + \text{total derivative}$$

which follows by using the second and third line of equation (8) twice and by 'partial integration'.

(*b*) From (*a*) we obtain the chain

$$\begin{aligned} \partial_x c_2^{(r)} &= 2\mathrm{i}\boldsymbol{b}^{(N-1)} \cdot \sigma_2 \boldsymbol{b}^{(n)} = 2\mathrm{i}\boldsymbol{b}^{(N-2)} \cdot \sigma_2 \boldsymbol{b}^{(n+1)} + \text{total derivative} = \ldots \\ &= 2\mathrm{i}\boldsymbol{b}^{(N-s-1)} \cdot \sigma_2 \boldsymbol{b}^{(n+s)} + \text{total derivative} = \ldots . \end{aligned}$$

Next we distinguish two cases: if $N+n-1=2p$ is even we arrive at $\boldsymbol{b}^{(p)} \cdot \sigma_2 \boldsymbol{b}^{(p)}$ which is zero; if $N+n-1=2q-1$ is odd, we arrive at $\boldsymbol{b}^{(q)} \cdot \sigma_2 \boldsymbol{b}^{(q-1)}$ which can be seen from (*a*) to be a total differential too.

3. The AKNS scheme—Lax pair construction

The second way to obtain the above-mentioned hierarchy follows the AKNS scheme, which is similar to construction of a Lax pair. Multiplying (1) by $\mathrm{i}H$ from the left yields

$$\partial_x \psi = X\psi \qquad X = \mathrm{i}H(\lambda - c_1) + \mathrm{i}a_1 - \boldsymbol{E} \cdot \boldsymbol{v}. \tag{13}$$

The time evolution should be given by a T operator according to

$$\partial_t \psi = T\psi. \tag{14}$$

(13) and (14) are integrable iff X and T fulfil

$$[\partial_x - X, \partial_t - T] = 0. \tag{15}$$

Next, expanding T in a power series in λ:

$$T = \sum_{n=0}^{N} \lambda^n (\mathrm{i}a_2^{(n)} - \mathrm{i}Hc_2^{(n)} - \boldsymbol{E} \cdot \boldsymbol{b}^{(n)}) \tag{16}$$

yields, after inserting (13) and (16) into (15), systems (8) and (9).

Remark. Special examples of the hierarchy have been treated in the literature. We may omit a_1 and $a_2^{(0)}$ since they decouple from the system.

For $N = 1$ we obtain as evolution equations

$$\partial_t c_1 = \partial_x c_2^{(0)} \qquad D_t \boldsymbol{v} = D_x \boldsymbol{v}. \tag{17}$$

A suitable gauge transformation allows us to put $c_1 = c_2^{(0)} = 0$.

For $N = 2$ a generalisation of the non-linear Schrödinger equation is obtained:

$$\partial_t c_1 = \partial_x c_2^{(0)} - \partial_x(\boldsymbol{v} \cdot \boldsymbol{v}) \qquad D_t \boldsymbol{v} = \mathrm{i}\sigma_2 D_x^2 \boldsymbol{v}. \tag{18}$$

Taking $c_1 = 0$ and $c_2^{(0)} = \boldsymbol{v} \cdot \boldsymbol{v}$ yields the standard form [9, 10].

For $N = 3$ we obtain a generalisation of a coupled system of MKdV equations

$$\begin{aligned} \partial_t c_1 &= \partial_x c_2^{(0)} - 2\mathrm{i}\partial_x(\boldsymbol{v} \cdot \sigma_2 D_x \boldsymbol{v}) \\ D_t \boldsymbol{v} &= 2D_x(\boldsymbol{v}(\boldsymbol{v} \cdot \boldsymbol{v})) - D_x^3 \boldsymbol{v} \end{aligned} \tag{19}$$

which goes over to the standard form if $c_2^{(0)} = 2\mathrm{i}\boldsymbol{v} \cdot \sigma_2 \partial_x \boldsymbol{v}$ [7].

Remark. Expansions like (16) also work for complex potentials. Real ones correspond to self-adjoint Dirac operators, non-real ones to the non-self-adjoint case. For the non-linear Schrödinger equation real potentials correspond to repulsion and pure imaginary ones to attraction [9, 10].

4. Lax pairs in terms of Kac–Moody generators

Recently it has been realised that infinite-dimensional integrable systems are related to infinite-dimensional Lie algebras [11, 12]. For certain systems such algebras have been identified as describing the symmetry of the system [13, 14]. In particular, the connection of Toda lattice systems to such algebras has been worked out.

Here we show that an expansion of Lax pairs in terms of generators of a Kac–Moody algebra leads to the hierarchy (8) and (9) too. We start from $\mathcal{A}_1$ and $\mathcal{A}_2$ expressed in terms of an element $g(t, x)$ belonging to the Kac–Moody group

$$\mathcal{A}_1 = -g(\partial_x g^{-1}) \qquad \mathcal{A}_2 = -g(\partial_t g^{-1}). \tag{20}$$

Consistency for (20) implies that

$$[\partial_x - \mathcal{A}_1, \partial_t - \mathcal{A}_2] = 0. \tag{21}$$

Next we expand $\mathcal{A}_1$ and $\mathcal{A}_2$ in terms of generators. We identify $\lambda^n H$, $\lambda^n E^1$ and $\lambda^n E^2$ in equations (13) and (16) with generators H_n, E^1_n and E^2_n which belong to the algebra (for a review see [15]):

$$\begin{aligned} &[H_m, H_n] = 0 && [E_m, E_n] = 0 \\ &[E^1_m, E^2_n] = 2\mathrm{i}H_{m+n} && [H_m, E^{\overset{1}{2}}_n] = \pm 2\mathrm{i} E^{\overset{2}{1}}_{n+m} \end{aligned} \tag{22}$$

and expand $\mathcal{A}_1$ and $\mathcal{A}_2$ as

$$\begin{aligned} \mathcal{A}_1 &= \mathrm{i}H_1 + \mathrm{i}1 a_1 - \mathrm{i}H_0 c_1 - \boldsymbol{E}_0 \cdot \boldsymbol{v} \\ \mathcal{A}_2 &= \mathrm{i}1 a_2^{(0)} - \sum_{n=0}^{N} (\mathrm{i}H_n c_2^{(n)} + \boldsymbol{E}_n \cdot \boldsymbol{b}^{(n)}). \end{aligned} \tag{23}$$

Inserting (23) into (21) again yields systems (8) and (9).

Remark. Since in expansion equation (23) only generators H_n and $\boldsymbol{E}_n$ with $n \geq 0$ occur, one obtains the same system whether one uses a central extension of the algebra or not.

Remark. Gauge transformations can now be expressed as

$$\begin{aligned} &\partial_{\substack{x\\t}} - \mathcal{A}_{\substack{1\\2}}(t, x) \to g_0(t, x)(\partial_{\substack{x\\t}} - \mathcal{A}_{\substack{1\\2}}(t, x))g_0^{-1}(t, x) \\ &g_0(t, x) = \exp(\mathrm{i}1\Lambda(t, x) - \mathrm{i}H_0\chi(t, x)) \end{aligned} \tag{24}$$

where $\{g_0\}$ form an Abelian subgroup of the Kac–Moody group.

Remark. Recently it has been realised that various realisations of elements of Kac–Moody algebras may also be of use for solving non-linear equations [11, 12]. A particular simple realisation starts from creation and annihilation operators A^r_n, $A^{r\dagger}_n$, $r = 1, 2$, $n \in Z$,

$$\{A^r_m, A^s_n\} = 0 \qquad \{A^{r\dagger}_m, A^s_n\} = \delta^{rs}\delta_{mn} \tag{25}$$

where finally

$$H_m = \sum_q \boldsymbol{A}^\dagger_{q-m} H \boldsymbol{A}_q \qquad E^{\overset{1}{2}}_m = \sum_q \boldsymbol{A}^\dagger_{q-m} E^{\overset{1}{2}} \boldsymbol{A}_q \tag{26}$$

fulfil the algebra (22). For the quantum mechanical vacuum $|0\rangle$ defined by $A^r_q|0\rangle = 0$ $\forall q$ H_m, E^1_m and E^2_m annihilate it for all m; in addition $\mathcal{A}_1$ and $\mathcal{A}_2$ leave the one-particle subspace invariant.

Remark. More interesting is the realisation where one takes the filled Dirac sea vacuum $|0\rangle$ with $A^r_m|0\rangle = 0$ for $m \geqslant 0$ and $A^{r\dagger}_m|0\rangle = 0$ for $m<0$; normal ordering of operators of (26) yields a central extension of the algebra (22) [12]. Since $\mathscr{A}_1$ and $\mathscr{A}_2$ again leave the one-particle subspace of the Hilbert space, built on $|0\rangle$, invariant, we may write

$$\partial_x|\psi\rangle = \mathscr{A}_1|\psi\rangle \qquad \partial_t|\psi\rangle = \mathscr{A}_2|\psi\rangle \tag{27}$$

and expand

$$|\psi\rangle = \sum_{n\geqslant 0} \boldsymbol{\phi}^+_n(t,x)\boldsymbol{A}^\dagger_n|0\rangle + \sum_{n<0} \boldsymbol{\phi}^-_n(t,x)\boldsymbol{A}_n|0\rangle. \tag{28}$$

Particle and antiparticle subspaces are separately left invariant so we may consider only the former. From (27) we get

$$\partial_x\boldsymbol{\phi}^+_n = \mathrm{i}H\boldsymbol{\phi}^+_{n+1} + (\mathrm{i}a_1 - \mathrm{i}Hc_1 - \boldsymbol{E}\cdot\boldsymbol{v})\boldsymbol{\phi}^+_n \qquad \forall n \geqslant 0. \tag{29}$$

Therefore if we put $\boldsymbol{\phi}^+_0$ equal to a spinor $\boldsymbol{\psi}(\lambda;t,x)$ which satisfies (13) we obtain

$$\boldsymbol{\phi}^+_n = \lambda^n\boldsymbol{\psi} \qquad \forall n \geqslant 0. \tag{30}$$

With that choice we obtain for $\mathscr{A}_1$ and $\mathscr{A}_2$

$$\begin{aligned} \mathscr{A}_1|\psi\rangle &= \sum_{n\geqslant 0} (X\boldsymbol{\phi}^+_n)\boldsymbol{A}^\dagger_n|0\rangle \\ \mathscr{A}_2|\psi\rangle &= \sum_{n\geqslant 0} (T\boldsymbol{\phi}^+_n)\boldsymbol{A}^\dagger_n|0\rangle. \end{aligned} \tag{31}$$

5. Conclusion

In this paper we have indicated three ways to obtain a hierarchy of non-linear equations which generalise well known systems of the literature. The most promising approach seems to us to be the connection to infinite-dimensional Lie algebras and we expect to obtain further results from it.

Let us finally mention that, besides the conventional Hamiltonian formulation [9, 10] where an ultralocal symplectic form is taken, there also exists an approach where a non-ultralocal form is taken and the evolution equations (9) for the Nth system are given by

$$\begin{aligned} \partial_t c_1 &= \{c_1, \mathscr{H}_N\}_{\mathrm{PB}} = \partial_x\, \partial\mathscr{H}_N/\partial c_1 \\ D_t\boldsymbol{v} &= \{\boldsymbol{v}, \mathscr{H}_N\}_{\mathrm{PB}} = D_x\, \partial\mathscr{H}_N/\partial\boldsymbol{v}. \end{aligned} \tag{32}$$

It is not difficult to construct the appropriate Hamiltonian $\mathscr{H}_N$. In a gauge $c_2^{(0)} = 0$ the covariant derivative D_t becomes the ordinary one. Further elaboration on these systems will be published elsewhere.

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